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CONVERGENCE OF REPRESENTATION AVERAGES AND OF CONVOLUTION POWERS

BY

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ABSTRACT

Let S be a locally compact (σ-compact) group or semi-group, and let $T(t)$ be a continuous representation of S by contractions in a Banach space X. For a regular probability μ on S, we study the convergence of the powers of the μ-average $Ux = \int T(t)x d\mu(t)$. Our main results for random walks on a group G are:

(i) if μ is adapted and strictly aperiodic, and generates a recurrent random walk, then $U^n(U-I)$ converges strongly to 0. In particular, the random walk is completely mixing.

(ii) If $\mu \times \mu$ is ergodic on $G \times G$, then for every unitary representation $T(.)$ in a Hilbert space, U^n converges strongly to the orthogonal projection on the space of common fixed points. These results are proved for semi-group representations, along with some other results (previously known only for groups) which do not assume ergodicity.

(iii) If μ is spread-out with support S, then $\|\mu^{n+K} - \mu^n\| \to 0$ if and only if $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j+K}}$.

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(iv) If G is in the class [SIN] and μ is adapted and strictly aperiodic, then for every representation by contractions in a uniformly convex or uniformly smooth Banach space, U^n converges strongly to a projection on the common fixed points. Other results for G in [SIN] are also obtained.

1. Introduction

Let $\mathcal{S}$ be a locally compact (σ-compact) semi-group (always assumed Hausdorff). For a regular probability μ on $\mathcal{S}$, the convolution operator $\mu * f(t) = \int f(ts)d\mu(s)$ is a Markov operator on $C(\mathcal{S})$, which is the average of the translation operators $\delta_s * f(t) = f(ts)$. When $\mathcal{S} = G$ is a locally compact group with right Haar measure λ, the representation $s \to \delta_s$ is continuous in $L_p(G,\lambda)$ $1 \le p < \infty$, $C_0(G)$ and $UCB_l(G)$.

Let X be a Banach space, and let T: $\mathcal{S} \to B(X)$ be a bounded operator of $\mathcal{S}$ (i.e., $T(st) = T(s)T(t)$, and $\sup_s \|T(s)\| < \infty$). The representation is called **continuous** if $t \to T(t)x$ is continuous for every $x \in X$ and **weakly continuous** if $f(t) = \langle x^*, T(t)x\rangle$ is continuous for $x \in X^*$ and $x \in X$. For groups, this implies (strong) continuity [HR, p.340]. For a regular probability μ on $\mathcal{S}$, the μ average $U_\mu x = \int T(t)xd\mu$ is defined in the strong operator topology for strongly continuous bounded representations. If X is reflexive and the representation is weakly continuous, then $U_\mu x$ is defined in the weak operator topology by $\langle x^*, U_\mu x\rangle = \int\langle x^*, T(t)x\rangle d\mu(t)$. It is easily checked that $U_{\mu*\nu} = U_\mu U_\nu$, with the convolution $\mu * \nu$ defined by $\mu * \nu(A) = \int\int 1_A(ts)d\mu(t)d\nu(s)$. Note that $U_\mu^* = \int T^*(t)d\mu(t)$ is always defined in the weak-$*$ topology of X^*.

There are two interesting problems in the study of the asymptotic behaviour of random walks on groups:

1. The mixing problem: If μ is an ergodic strictly aperiodic probability (i.e., G is the smallest closed normal subgroup, a class of which contains the support of μ), is the random walk generated by μ completely mixing (i.e. $\|\mu^n * f\|_1 \to 0$ for $\int f d\lambda = 0$)? We give an affirmative answer if μ generates a *recurrent* random walk.

2. The "concentration function" problem: If G is non-compact, and μ is **adapted** (the closed group generated by its support is G) and strictly aperi-

odic, do we have $\|\mu^n * f\|_\infty \to 0$ for every $f \in C_0(G)$? We obtain an affirmative answer if $\mu \times \mu$ is ergodic.

The results are obtained by studying the convergence of the iterates of the μ-average of a representation of a semi-group $\mathcal{S}$, under various assumptions on μ. We also extend some of the results of [DL$_3$] from group representations (in uniformly convex spaces) to semi-group representations.

The "concentration function" problem is given a positive solution if there exists an open set V with compact closure such that $t^{-1}Vt = V$ for every $t \in G$. This is obtained via the study of representations of groups in the class *[SIN]*.

Completing the work in [HoM], Willis [Wi] has recently proved the "concentration function" problem for μ **irreducible** (the closed semi-group generated by its support is G) on *any* locally compact group. The problem for μ adapted and strictly aperiodic is still unsolved for arbitrary G.

2. Strong convergence of U^n

One of the main problems in studying the asymptotic behaviour of random walks on *groups* and of convolution powers is: *If μ is an ergodic strictly aperiodic probability on G, is it completely mixing?* (complete mixing means that

$$\|\mu^n * f\|_1 \to 0 \quad \text{for } f \in L_1(G,\lambda) \quad \text{with } \int f d\lambda = 0.$$

See [K, p.254]). The answer is affirmative in the following cases:

(1) G is compact — the Ito–Kawada theorem (e.g., [Ro, p.153] or [DL$_3$, cor. 3.2]).

(2) G is abelian [KeMa], [St] (see also [F$_1$], [DL$_1$], [DL$_2$], [L$_2$]).

(3) For some n, μ^n and μ^{n+1} are not mutually singular [F$_2$].

(4) μ is spread out (for some n, μ^n is not singular with respect to the Haar measure λ) [G]. (In particular if G is discrete.)

In this section we study the problem via general results on representations, and we do it in the context of semi-groups. For abelian semigroups, see [L$_1$].

Let $\mathcal{S}$ be a semi-group, and $T(.)$ a continuous bounded operator representation in a Banach space. We denote $F = \{y\colon T(t)y = y \text{ for } t \in \mathcal{S}\}$, and $N = \operatorname{clm}\bigcup_{t\in\mathcal{S}}(I - T(t))X$. We define similarly F_* and N_* for $\{T^*(t)\}$. The Hahn–Banach theorem yields that $z \in N \Leftrightarrow \langle y^*, z\rangle = 0$ for every $y^* \in F_*$.

Definition: A probability μ on a locally compact σ-compact semi-group $\mathcal{S}$ is called **ergodic** if the only $g \in C(\mathcal{S})$, satisfying $\int g(st)d\mu(s) = g(t)$ for every $t \in$S, are the constant functions.

We define $g * \mu = \int g(st)d\mu(s)$. Let G be a locally compact group, and $f \in C(G)$. Define $\check{f}(t) = f(t^{-1})$. Then

$$(2.1)\quad \check{\mu}*f(t) = \int f(ts)d\check{\mu}(s) = \int f(ts^{-1})d\mu(s) = \int \check{f}(st^{-1})d\mu(s) = \check{f}*\mu(t^{-1}).$$

This shows that $\check{\mu} * f \geq f$ if and only if $\check{f} * \mu \geq \check{f}$, and μ is ergodic in that definition if and only if $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \mu^j$ is an ergodic sequence in the sense of [LW].

Lemma 2.1: *Let μ be an ergodic probability on a locally compact semi-group $\mathcal{S}$, and let $T(t)$ be a bounded continuous representation of $\mathcal{S}$ in a Banach space X. Then:*

(i) $U_\mu^* y^* = y^* \Leftrightarrow y^* \in F_*$.

(ii) $\overline{(I - U_\mu)X} = N$.

(iii) $F \cap N = \{0\}$.

(iv) *For every $x \in X$, $\overline{\text{co}}\{T(t)x\colon t \in \mathcal{S}\} \cap F$ has at most one point.*

Proof: We denote U_μ by U. We may assume $\|T(t)\| \leq 1$.

(i) Let $U^*y^* = y^*$. For $x \in X$ define $g(t) = \langle y^*, T(t)x\rangle$. Then

$$g * \mu(t) = \int g(st)d\mu(s) = \int \langle y^*, T(s)T(t)x\rangle d\mu(s)$$
$$= \langle y^*, UT(t)x\rangle = \langle y^*, T(t)x\rangle = g(t).$$

Hence $g(t)$ is constant by ergodicity of μ, and integration with respect to μ yields $g(t) = \langle y^*, x\rangle$. Thus $\langle T^*(t)y^*, x\rangle = \langle y^*, x\rangle$ for every $t \in \mathcal{S}$ and $x \in X$, so $y^* \in F_*$.

(ii) Obviously $\overline{(I - U)X} \subset N$. The Hahn–Banach theorem and (i) yield equality.

(iii) $F \cap N \subset \{y\colon Uy = y\} \cap N = \{y\colon Uy = y\} \cap \overline{(I - U)X} = \{0\}$.

(iv) Let $t_1 \neq y_2$ be in $C = \overline{\text{co}}\{T(t)x\colon t \in \mathcal{S}\}$. Then for every $y^* \in F_*$ we have $\langle y^*, y_1\rangle = \langle y^*, y_2\rangle$, since y^* is constant on C. Thus $0 \neq y_1 - y_2 \in F \cap N$, a contradiction. ∎

Remarks: (1) Property (ii) is equivalent to saying that $\frac{1}{N}\sum_{j=1}^{N} U^j$ is a *right-ergodic* sequence for $\{T(t): t \in \mathcal{S}\}$; see [K, pp. 75–81]. For any μ (ergodic or not), we have (i)$\Leftrightarrow$ (ii).

(2) Let X be the dual of the example of [AlBi], and $\mathcal{S}$ the dual of the (two-point) semi-group they construct. Since X^* is now uniformly convex, (i) holds, and $\mu = \frac{1}{2}(\delta_1 + \delta_2)$ is ergodic.

However, there exists $y \in X$, $Uy = y$, with $y \notin F$. This shows that right ergodicity is *not* sufficient for Eberlein's theorem [E] (see [K, p.76]), and answers negatively the question of [K, p.78].

(3) If the only functions $g \in C(\mathcal{S})$ satisfying $\mu * g = g$ are constants, a similar proof shows $Uy = y \Leftrightarrow y \in F$ (weak continuity is sufficient for this proof as well as for Lemma 2.1(i)). Note that for a *group*, this is ergodicity of $\check{\mu}$, by (2.1). However, μ may be ergodic with $\check{\mu}$ not ergodic [G], [D$_2$], [D$_3$].

(4) Ergodic probabilities exist on a locally compact σ-compact group if it is *amenable* ([KaV], [Ros]), and only if ([A],[Fu]).

(5) In an Abelian *group*, μ is ergodic if and only if it is adapted.

Definition: μ on the semi-group is **topologically recurrent** if we have

$$g \in C(\mathcal{S}), \quad g * \mu \geq g \Rightarrow g * \mu = g. \tag{2.2}$$

Remarks: (1) If μ is adapted on a group, this is equivalent to recurrence of the random walk generated by μ (e.g., [Re]).

(2) If μ is recurrent, so is μ^j for every $j > 1$.

(3) Recurrence in semi-groups is studied in [MTs]; see also [Ro].

Definition: Let A, B be subsets of a semigroup $\mathcal{S}$. We define

$$A^{-1}B = \{ t \in \mathcal{S}: \exists a \in A \text{ with } at \in B\},$$
$$BA^{-1} = \{ t \in \mathcal{S}: \exists a \in A \text{ with } ta \in B\}.$$

THEOREM 2.2: *Let μ be a topologically recurrent ergodic probability on a locally compact (σ-compact) semi-group $\mathcal{S}$, with support S, such that the closed semi-group generated by $\bigcup_{j=1}^{\infty}(S^j)^{-1}S^j$ is $\mathcal{S}$. For a bounded continuous representation of $\mathcal{S}$ in a Banach space X, we have $\lim \|(U_\mu)^n x\| = 0$ for $x \in N = \operatorname{clm}\bigcup_{t\in\mathcal{S}}(I - T(t))X$, so, in particular, $\|U^n(I - U)y\| \to 0$ for $y \in X$.*

Proof: By taking an equivalent norm, we may assume $\|T(t)\| \leq 1$ for $t \in \mathcal{S}$. Denote U_μ by U. Then $\|U\| \leq 1$, so $\|U^n x\|$ converges for every $x \in X$.

STEP 1: For every $x \in X$, $\lim_{n\to\infty} \|U^n T(t)x\|$ is independent of $t \in \mathcal{S}$.

Fix $x \in X$, and define $h_n(t) = \|U^n T(t)x\|$. Then

$$\begin{aligned} h_n * \mu(t) &= \int h_n(st)d\mu(s) = \int \|U^n T(st)x\| d\mu(s) \\ &\geq \| \int U^n T(s)T(t)x d\mu(s)\| = \|U^{n+1}T(t)x\| = h_{n+1}(t). \end{aligned}$$

Since $\|U\| \leq 1$, $h_n(t)$ is decreasing, and $h(t) = \lim h_n(t)$ is well-defined, and satisfies $h * \mu \geq h$. By equicontinuity of $\{h_n\}$, $h(t)$ is continuous. Hence, by (2.2), $h * \mu = h$, and by ergodicity, $h(t)$ is a constant function, so our claim is proved.

It follows that if $y = T(t_0)x$, then

$$\lim \|U^n T(t)y\| = \lim \|U^n T(tt_0)x\| = \lim \|U^n T(t_0)x\| = \lim \|U^n y\|.$$

We fix x satisfying:

$$(*) \qquad \lim_n \|U^n T(t)x\| = \lim \|U^n x\| \quad \text{for every } t \in \mathcal{S}.$$

STEP 2: If $\mu(A) > 0$, then $\lim_{n\to\infty} \|\frac{1}{\mu(A)} \int_A U^n T(t)x d\mu(t)\| = \lim_{n\to\infty} \|U^n x\|$.

Let $c = \lim \|U^n x\|$. By $(*)$, $\lim \|U^n T(t)x\| = c$ for every $t \in \mathcal{S}$. We denote $\frac{1}{\mu(A)} \int_A T(t)x d\mu(t)$ by $U_A x$. Then $\lim \|U^n U_A x\|$ exists since $\|U\| \leq 1$, and, by Lebesgue's theorem,

$$\lim \|U^n U_A x\| \leq \lim \frac{1}{\mu(A)} \int_A \|U^n T(t)x\| d\mu(t) = c.$$

We use this inequality also for A^c and obtain

$$\begin{aligned} c = \quad & \lim \|U^{n+1}x\| = \lim \|\mu(A)U^n U_A x + \mu(A^c)U^n U_{A^c} x\| \\ & \leq \mu(A) \lim \|U^n U_A x\| + \mu(A^c) \lim \|U^n U_{A^c} x\| \leq c \end{aligned}$$

Hence we have $\|U^n U_A x\| \to c$.

STEP 3: Let $A \cap B = \emptyset$, $\mu(A)\mu(B) > 0$. Then

$$\lim \|U^n U_A x + U^n U_B x\| = 2 \lim \|U^n x\|.$$

We may assume $\mu(A) \leq \mu(B)$. Define

$$y_n = \frac{1}{\mu(A \cup B)} \int_A U^n T(t)x d\mu(t) = \frac{\mu(A)}{\mu(A \cup B)} U^n U_A x$$

and

$$z_n = \frac{\mu(B)}{\mu(A \cup B)} U^n U_B x.$$

By step 2 we have $\lim \|y_n\| = c\mu(A)/\mu(A \cup B)$ and $\lim \|z_n\| = c\mu(B)/\mu(A \cup B)$, and also $\|y_n + z_n\| = \|U^n U_{(A\cup B)} x\| \downarrow c$.

Of course, we have to prove only for $c > 0$.

Let $\delta > 0$. Fix N such that for $n > N$ we have

$$\|y_n\| < \frac{\mu(A)c(1+\delta)}{\mu(A \cup B)}; \quad \|z_n\| < \frac{\mu(B)c(1+\delta)}{\mu(A \cup B)}; \quad \|y_n + z_n\| \geq c.$$

For $\alpha = \mu(A)/\mu(B)$ (we assumed $\alpha \leq 1$) we have

$$\begin{aligned}\|y_n + \alpha z_n\| = \|y_n + z_n - (1-\alpha)z_n\| &\geq \|y_n + z_n\| - (1-\alpha)\|z_n\| \\ &\geq c - \frac{\mu(B) - \mu(A)}{\mu(B)} \frac{\mu(B)c(1+\delta)}{\mu(A \cup B)}.\end{aligned}$$

Hence

$$\begin{aligned}\|U^n U_A x + U^n U_B x\| &= \frac{\mu(A \cup B)}{\mu(A)} \|y_n + \alpha z_n\| \\ &\geq [\mu(A)]^{-1} \{c[\mu(A) + \mu(B)] - c(1+\delta)[\mu(B) - \mu(A)]\} \\ &= 2c - \delta c[\mu(B)/\mu(A) - 1].\end{aligned}$$

Letting $\delta \to 0$ we have (using step 2)

$$2c \leq \lim \|U^n(U_A x + U_B x)\| \leq \lim \|U^n U_A x\| + \lim \|U^n U_B x\| = 2c.$$

STEP 4: *$\lim \|U^n T(t_1)x + U^n T(t_2)x\| = 2 \lim \|U^n x\|$ for $t_1, t_2 \in S$.*

By strong continuity, for $\varepsilon > 0$ there exist disjoint open sets A and B, $t_1 \in A$ and $t_2 \in B$, such that $\|T(t)x - T(t_1)x\| < \varepsilon$ for $t \in A$ and $\|T(t)x - T(t_2)x\| < \varepsilon$ for $t \in B$. Step 3 will yield the claim.

Proof of the Theorem: By step 1, $\lim \|U^n T(t)x\|$ is a constant, and we denote it by $c(x)$. Using again step 1, we obtain

$$c(T(s)x) = \lim_{n\to\infty} \|U^n T(t)T(s)x\| = \lim \|U^n T(ts)x\| = c(x) = \lim \|U^n T(s)x\|.$$

Hence assumption (*) holds with $T(s)x$ instead of x, for *any* $x \in X$. Thus, we may substitute in step 4 $T(s)x$ for x, and since $c(T(s)x) = c(x)$, we obtain, for $s \in \mathcal{S}$, $x \in X$, and $t_1, t_2 \in S$ that

$$\lim_{n\to\infty} \|U^n T(t_1)T(s)x + U^n T(t_2)T(s)x\| = 2c(x).$$

Let $t \in S^{-1}S$. Then there are $t_1, t_2 \in S$ with $t_1 t = t_2$. Use these t_1, t_2 and put $s = t$ in the above to obtain

$$\begin{aligned} c(x + T(t)x) &= \lim \|U^n T(t_2)(x + T(t)x)\| \\ &= \lim \|U^n T(t_1)T(t)x + U^n T(t_2)T(t)x\| = 2c(x). \end{aligned}$$

Hence,

$$c\Big(\frac{I + T(t)}{2}x\Big) = c(x) \quad \text{for } t \in S^{-1}S.$$

Since also U^j obviously satisfies the conclusion of step 1 (with same limits $c(x)$), we have for $t \in (S^j)^{-1}S^j$ and $x \in X$ that

$$c\Big(\frac{I + T(t)}{2}x\Big) = c(x).$$

By induction we obtain

$$c((\frac{I + T(t)}{2})^k x) = c(x) \quad \text{for } x \in X \quad \text{and} \quad t \in \bigcup_j (S^j)^{-1}S^j.$$

Fix $t \in \bigcup (S^j)^{-1}S^j$ and let $x = (I - T(t))y$. Computation of binomial coefficients shows that

$$\|(\frac{I + T(t)}{2})^k (I - T(t))\| \xrightarrow[k \to \infty]{} 0.$$

Hence

$$c(x) = c((I - T(t))y) = c\Big(\big(\frac{I + T(t)}{2}\big)^k (I - T(t))y\Big) < \varepsilon$$

for large k, so $c(x) = 0$ for $x = (I - T(t))y$. Thus, for $t \in \bigcup (S^j)^{-1}S^j$ we obtain $\lim_{n\to\infty} \|U^n T(s)(I - T(t))y\| = 0$ for every $s \in \mathcal{S}$ and $y \in X$.

Fix $x \in X$ and define

$$\mathcal{S}' = \{\, t \in \mathcal{S} \colon \|U^n T(s)(I - T(t))x\| \to 0 \text{ for every } s \in \mathcal{S}\}$$

(i.e., $\mathcal{S}' = \{\, t\colon c((I - T(t))x) = 0\}$). By what we have proved (putting x instead of y), $\bigcup (S^j)^{-1}S^j \subset \mathcal{S}'$, and clearly (by continuity of the representation) $\mathcal{S}'$ is closed. We show it is a semi-group: let $t_1, t_2 \in \mathcal{S}'$. Then, for any $s \in \mathcal{S}$, we have

$$U^n T(s)(I - T(t_1 t_2))x = U^n T(s)(I - T(t_1))x + U^n T(s)T(t_1)(I - T(t_2))x$$

which shows $t_1, t_2 \in \mathcal{S}'$. By our assumption on S, we have $\mathcal{S}' = \mathcal{S}$. Using Lebesgue's theorem, we obtain for any $t \in \mathcal{S}$ that

$$\|U^{n+1}(I - T(t))x\| \le \int \|U^n T(s)(I - T(t))x\| d\mu(s) \to 0$$

and the theorem is proved. ∎

Remarks: (1) $\mathcal{S} = \{a,b\}$ with $a^2 = ab = a$ and $b^2 = ba = b$, with $\mu\{a\}\mu\{b\} > 0$, satisfies the assumptions of the theorem. However, the dual of the example of [AlBi], discussed above, shows that even if $U^n x$ converges, the limit need not be in F (this also shows that the identification of the limit in [K, p.222] as being in F requires *additional* assumptions — e.g. $\mathcal{S}$ is a group [RN]).

(2) The recurrence is used only in step 1. If we can show, for a representation by contractions, that for every $x \in X$ $\lim \|U^n T(t)x\|$ is constant (independent of t), then the rest of the proof is applicable. (This is the case for the translation operators in $L_1(G)$ for G an Abelian group.)

(3) If G is a locally compact group, strict aperiodicity of adapted μ is equivalent to the fact that the closed (semi) group generated by $\bigcup_{j=1}^{\infty}(S^{-j}S^j \bigcup S^jS^{-j})$ is all of G [DL$_3$]. For a group put, in step 4, $x = T(t_2^{-1})x$, $t_2 \in S$, to obtain

$$c\Big(\frac{I+T(t)}{2}x\Big) = c(x)$$

also for $t \in SS^{-1}$. Our proof then applies to this case. Hence the Abelian group result is included.

(4) It can be shown that a proof of theorem 4 of [L$_1$], which deals with S Abelian, is also included in our proof.

COROLLARY 2.3: *Let G be a locally compact σ-compact group with right Haar measure λ, and let μ be an ergodic strictly aperiodic probability. If μ is recurrent, then $\|\mu^n * f\|_1 \to 0$ for every $f \in L_1(G,\lambda)$ with $\int f d\lambda = 0$ (i.e., μ is completely mixing).*

Proof: The recurrence of μ implies irreducibility, and then the strict aperiodicity is equivalent to the condition imposed on S in the theorem, by [DL$_3$] (see also Remark (3) above). We apply the theorem to the representation $T(t)f = \delta_t * f$ for $f \in L_1(G,\lambda)$, noting that $N = \{ f \in L_1(G,\lambda)\colon \int f d\lambda = 0\}$. ∎

Remarks: (1) The existence of a recurrent ergodic random walk on a group G implies amenability and *unimodularity*. See [GuKeRoy], where such groups are studied.

(2) In [DL$_4$] it is shown how to transfer the complete mixing in $L_1(G)$ to all representations. However, it is not needed here, since the theorem was proved directly for representations, and yields the following.

COROLLARY 2.4: *Let G be a locally compact group and μ a recurrent ergodic strictly aperiodic probability. For a bounded continuous representation of G in a Banach space X we have $\{y\colon U_\mu y = y\} = F$, and U_μ^n converges strongly on $F \oplus N$ to the corresponding projection on F.*

Proof: Since μ is recurrent and ergodic, so is $\check{\mu}$, and $U_\mu y = y \Rightarrow y \in F$ is proved as in Lemma 2.1 (i). We can now apply Theorem 2.2. ∎

Remarks: (1) If X is reflexive, for bounded groups the decomposition $X = F \oplus N$ holds *without* requiring any recurrent ergodic probability, by [RN].

(2) If X is reflexive and the group G has a recurrent ergodic probability, then $X = F \oplus N$ follows from Lemma 2.1(i) and the proof of Corollary 2.4, using the ergodic splitting induced by U_μ.

(3) The decomposition $X = F \oplus N$ was studied by Eberlein [E].

COROLLARY 2.5: *Under the assumptions of Corollary 2.4, if G is non-compact, then*

(i) $\|\mu^n * f\|_2 \to 0$ *for* $f \in L_2(G, \lambda)$.

(ii) $\|\mu^n * f\|_\infty \to 0$ *for* $f \in C_0(G)$.

Remark: It was noted in [DL3] that the above corollary is true without the strict aperiodicity assumption (for μ recurrent ergodic), using a general result for recurrent ergodic Markov operators.

It is well-known (e.g., [DL1]) that if μ is completely mixing on a group, then μ must be strictly aperiodic (and ergodic). See also Proposition 3.3. An example in [G] shows that we may have μ adapted and strictly aperiodic without $U^n(I-U)$ converging strongly to 0 (and $U^n(I-U^k)$ does not converge to zero, as $n \to \infty$, for no $k \geq 1$). Thus, we need a stronger algebraic condition, which can yield convergence results in the adapted non-ergodic case.

LEMMA 2.6: *Let G be a locally compact σ-compact group, μ a probability on G with support S, ν a probability with support S_ν. If*

$$\lim_{n\to\infty} \|\mu^n * (f - \nu * f)\|_1 < 2\|f\|_1 \quad \text{for every } f \in C(G) \text{ with compact support,}$$

then $e \in \overline{\bigcup_{j=0}^\infty S^{-j} S^j S_\nu}$.

Proof: Assume $e \notin \overline{\bigcup_{j=0}^\infty S^{-j} S^j S_\nu}$. Then there exists a neighbourhood A of e with $A \cap \bigcup_{j=0}^\infty S^{-j} S^j S_\nu = \emptyset$. Let B be a compact neighbourhood of e with $BB^{-1} \subset A$. Then $S^j B \cap S^j S_\nu B = \emptyset$ for every $j \geq 0$.

Take $0 \leq f \in C(G)$ supported on B with $\int f d\lambda = 1$, and let η be defined by $\frac{d\eta}{d\lambda} = f$. Then $\mu^j * \eta(S^j B) = 1$, $\mu^j * \nu * \eta(S^j S_\nu B) = 1$. Hence for every j,

$$2 = \|\mu^j * \eta - \mu^j * \nu * \eta\| = \|\mu^j * ((\Delta \check{f}) - \nu(\Delta \check{f}))\|_1$$

(where Δ is the modular function. See [LW]). But this contradicts the given convergence. ∎

COROLLARY 2.7: *Let μ be a probability with support S on a locally compact σ-compact group G.*

(i) *If μ is completely mixing, then $G = \overline{\bigcup_{j=0}^{\infty} S^{-j} S^j}$.*

(ii) *If μ is ergodic, then $G = \overline{\bigcup_{j=0}^{\infty} \bigcup_{k=0}^{\infty} S^{-j} S^k}$.*

(iii) *If, for some $k \geq 1$, we have $\lim_{n\to\infty} \|\mu^n * (f - \mu^k * f)\|_1 = 0$ for every $f \in L_1(G, \lambda)$, then $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j} S^{j+k}}$.*

Proof: (i) From Lemma 2.1(ii) and ergodicity of μ, we obtain

$$\lim_{n\to\infty} \|\mu^n * (f - \delta_t * f)\|_1 = 0 \quad \text{for every } f \in L_1(G,\lambda) \text{ and } t \in G.$$

Hence, by the previous lemma, $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j} S^j t}$, for every $t \in G$, which yields the assertion.

(ii) Let $\mu' = \frac{1}{2}(\delta_e + \mu)$. Then μ' has support $S_1 = S \cup \{e\}$ and is completely mixing. Since $S_1^j = \bigcup_{k=0}^{j} S^k$, with $S^0 = \{e\}$, we obtain the result by applying (i) to μ'.

(iii) is immediate from the previous lemma. ∎

Remarks: (1) If G is Abelian, $\overline{\bigcup_{j=0}^{\infty} S^{-j} S^j}$ is obviously a closed semi-group, closed under inversion, so it is the closed subgroup generated by $S^{-1}S$. Hence, the condition in (i) is equivalent to strict aperiodicity in the Abelian case.

(2) If μ is adapted and $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j} S^{j+1}}$, then μ is strictly aperiodic: Let H be the smallest closed normal subgroup with $S \subset tH$ for some $t \in G$. Since H is normal, $S^j \subset t^j H$, and

$$S^{-j} S^{j+1} \subset H t^{-j} t^{j+1} H = HtH = tH.$$

Hence the assumption implies $e \in tH$, so $t^{-1} \in H$, and thus $S \subset H$. Since μ is adapted, $H = G$.

(3) If $G = \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j}}$, then μ is adapted and strictly aperiodic: The closed subgroup generated by S obviously contains all $S^{-j}S^{j}$, hence is G. Now $GS = G$ yields $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j+1}}$, and by the previous remark, μ is strictly aperiodic.

(4) If $G = \overline{\bigcup_{j=0}^{\infty} \bigcup_{k=0}^{\infty} S^{-j}S^{k}}$, then μ is necessarily adapted.

(5) For G Abelian, μ is strictly aperiodic if and only if it is adapted and $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j+1}}$. This is because strict aperiodicity is equivalent to complete mixing in Abelian groups, and we can use Corollary 2.7 (iii), with $k = 1$.

We shall prove below the converse of Corollary 2.7 (iii) for μ spread-out, and for general μ in a certain class of groups (which contains all Abelian, compact and discrete groups).

LEMMA 2.8: *Let μ be a probability on a semi-group $\mathcal{S}$, and let $T(t)$ be a continuous representation of $\mathcal{S}$ by contractions in a Banach space X. Then for any $j \geq 0$, $k \geq 0$ and $x \in X$, we have*

(i) $\lim_{n\to\infty} \|T(t)U^n x + T(s)U^{n+k}x\| = 2\lim\|U^n x\|$ *in* $\mu^{k+j}(t)\times\mu^j(s)$ *measure.*

If $\mathcal{S}$ is a group, then we have also

(ii) $\lim_{n\to\infty} \|T(t)U^n x + U^{n+k}x\| = 2\lim\|U^n x\|$ *in* $\check{\mu}^j * \mu^{k+j}$ *measure.*

Proof: Denote $c(x) = \lim\|U^n x\|$ (which exists since $\|U\| \leq 1$). Then

$$\begin{aligned}2\|U^{n+k+j}x\| &= \left\| \int\int [T(t)U^n x + T(s)U^{n+k}x] d\mu^{k+j}(t) d\mu^j(s) \right\| \\ &\leq \int\int \|T(t)U^n x + T(s)U^{n+k}x\| d\mu^{k+j}(t) d\mu^j(s) \leq 2\|U^n x\|.\end{aligned}$$

Hence $\int\int [2\|U^n x\| - \|T(t)U^n x + T(s)U^{n+k}x\|] d\mu^{k+j}(t) d\mu^j(s) \to 0$ with non-negative integrand, which proves (i).

If $\mathcal{S}$ is a group (and we then assume $T(e) = I$), then $T(t)$ is an isometry, so

$$\begin{aligned}&\int [2\|U^n x\| - \|T(r)U^n x + U^{n+k}x\|] d(\check{\mu}^j * \mu^{k+j})(r) \\ &= \int\int [2\|U^n x\| - \|T(s^{-1}t)U^n x + U^{n+k}x\|] d\mu^j(s) d\mu^{k+j}(t) \\ &= \int\int [2\|U^n x\| - \|T(t)U^n x + T(s)U^{n+k}x\|] d\mu^j(s) d\mu^{k+j}(t)\end{aligned}$$

which converges to 0 by (i), and (ii) follows. ∎

For our next result we need a certain technical property of G. It is customary to denote by [*SIN*] the class of all locally compact groups such that there is a base of compact neighbourhoods of the identity, each invariant under all the inner automorphisms $\varphi_a(t) = a^{-1}ta$. This is a classical class of locally compact groups, and has been extensively studied. See [GrMo] and [Pa] (where many additional references are given).

THEOREM 2.9: *Let G be a locally compact group in [SIN]. Let μ be a probability on G, with support S, and let $K \geq 1$. Then the following are equivalent:*

(i) *for any bounded continuous representation of G in a Banach space X,*

$$\lim_{n\to\infty} \|U^n(I - U^K)x\| = 0 \quad \text{for every } x \in X.$$

(ii) $\lim_{n\to\infty} \|\mu^n * (f - \mu^K * f)\| < 2\|f\|_1$ *for $f \in C(G)$ with compact support.*

(iii) $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j+K}}$.

Proof: (i)⇒(ii). We apply (i) to the canonical representation by translations in $L_1(G,\lambda)$.

(ii)⇒(iii) follows from Lemma 2.6.

(iii)⇒(i): We may assume $\|T(t)\| \leq 1$ for $t \in G$, so each $T(t)$ is an isometry. Fix $x \in X$, and let $\varepsilon > 0$. By continuity, there exists a neighbourhood A of e such that $\|T(t)x - x\| < \varepsilon$ for $t \in A$, and since $G \in [SIN]$, we may take A to be invariant under the inner automorphisms. Hence, if $t^{-1}t' \in A$, we have that $(s^{-1}ts)^{-1}(s^{-1}t's) = s^{-1}(t^{-1}t')s \in A$ for every $s \in G$, hence

$$\|T(ts)x - T(t's)x\| = \|T(s^{-1}ts)x - T(s^{-1}t's)x\| < \varepsilon.$$

Now if $t^{-1}t' \in A$ we have also

$$\begin{aligned}\|T(ts)U^m x - T(t's)U^m x\| &= \| \int [T(ts)T(s')x - T(t's)T(s')x]d\mu^m(s')\| \\ &\leq \int \|T(t)T(ss')x - T(t')T(ss')x\|d\mu^m(s') < \varepsilon.\end{aligned}$$

Thus, if $y = U^m x$, then $t^{-1}t' \in A$ implies $\|T(ts)y - T(t's)y\| < \varepsilon$ for every $s \in G$. Hence the same holds also for $y \in C = \operatorname{Conv}\{U^m x : m \geq 0\}$.

Define $g_{n,k,y}(t) = \|T(t)U^n y + U^{n+k}y\|$. We want to show that $\{g_{n,k,y} : n \geq 0,\ k \geq 0,\ y \in C\}$ is equicontinuous. Fix $\varepsilon > 0$. Let A be the neighbourhood as before. For $t^{-1}t' \in A$ we have

$$\begin{aligned}|g_{n,k,y}(t) - g_{n,k,y}(t')| &\leq \|T(t)U^n y - T(t')U^n y\| \\ &\leq \int \|T(t)T(s)y - T(t')T(s)y\|d\mu^n(s) < \varepsilon.\end{aligned}$$

Hence $\{g_{n,k,y}\}$ is equicontinuous.

We denote $c(y) = \lim \|U^n y\|$.

Fix $t_0 \in \overline{S^{-j}S^{j+k}}$. We show that $\|T(t_0)U^n y + U^{n+k}y\| \to 2c(y)$ for $y \in C$. Let $\lim_{n_i} \|T(t_0)U^{n_i}y + U^{n_i+k}y\| = \alpha$. By Lemma 2.8 (ii), $\|T(t)U^n y + U^{n+k}y\| \to 2c(y)$ in $\check{\mu}^j * \mu^{j+k}$ measure, so for a subsequence of $\{n_i\}$ (still denoted $\{n_i\}$) we have $\check{\mu}^j * \mu^{j+k}$ a.e. convergence. By the equicontinuity $\lim_{n\to\infty} \|T(t)U^{n_i}y + U^{n_i+k}y\|$ is continuous on $\operatorname{supp}(\check{\mu}^j * \mu^{j+k})$, so $\alpha = 2c(y)$.

Fix $\varepsilon > 0$. By the equicontinuity, there exists an open set A containing e such that $t \in A$ implies $|g_{n,k,y}(t) - g_{n,k,y}(e)| < \varepsilon$ for every $n, k \geq 0$ and $y \in C$.

By our assumption, $e \in \overline{\bigcup_{j=0}^{\infty} S^{-j}S^{j+K}}$. Hence there exists $j \geq 0$ with $\check{\mu}^j * \mu^{j+K}(A) > 0$, so there is $t_0 \in A \cap \overline{S^{-j}S^{j+K}} \neq \emptyset$, and by the above, we have, for any $y \in C$ and $n \geq 1$

$$\begin{aligned}\|U^n y + U^{n+K}y\| &= g_{n,K,y}(e) \geq g_{n,K,y}(t_0) - \varepsilon \\ &= \|T(t_0)U^n y + U^{n+K}y\| - \varepsilon \xrightarrow[n\to\infty]{} 2c(y) - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\|U^n \frac{1}{2}(I + U^K)y\| \xrightarrow[n\to\infty]{} c(y)$. Hence

$$c\Big(\frac{I + U^K}{2} y\Big) = c(y),$$

and iterating,

$$c\Big(\Big(\frac{I + U^K}{2}\Big)^m x\Big) = c(x).$$

Since x was arbitrary, put $(I - U^K)x$ for x. Since

$$\|\Big(\frac{I + U^K}{2}\Big)^m (I - U^K)\| \xrightarrow[m\to\infty]{} 0$$

by computation of binomial coefficients, we have

$$\begin{aligned}\lim_{n\to\infty} \|U^n(I - U^K)x\| &= c((I - U^K)x) = c\Big(\Big(\frac{I + U^K}{2}\Big)^m (I - U^K)x\Big) \\ &\leq \|\Big(\frac{I + U^K}{2}\Big)^m (I - U^K)x\| \xrightarrow[m\to\infty]{} 0. \quad ∎\end{aligned}$$

Remarks: (1) The whole set C was not really used in this theorem, and we could have put x instead of $y \in C$. The general case will be needed later.

(2) If $e \in S^K$ (e.g., S is symmetric, yielding $e \in S^2$) (iii) holds, and we obtain $\|U^n(I - U^K)x\| \to 0$ for every $x \in X$. In particular, $e \in S \Rightarrow \|U^n(I - U)x\| \to 0$ for every $x \in X$.

(3) For G discrete and $e \in S^K$ (or, more generally, $\mu^K(\{e\}) > 0$), the theorem follows immediately from [F_2]. Hence the interest is in the non-discrete case.

(4) If μ is also ergodic and $K = 1$, we obtain that μ is completely mixing.

COROLLARY 2.10: *Under the assumptions of Theorem 2.9, if X is reflexive, and μ is adapted and strictly aperiodic, then $U^n x$ converges strongly to a projection P on F, with $P(N) = \{0\}$.*

Proof: By reflexivity $X = \{y\colon Uy = y\} \oplus \overline{(I-U)X}$. Since μ is adapted, $Uy = y \Rightarrow y \in F$, by [RN] (for a more detailed proof, see [T_1, p.28], or [T_2]). Also by [RN], we have $X = F \oplus N$ (these results of [RN] are *not* true for general *semi*-groups, e.g., [AlBi]). Hence necessarily $\overline{(I-U)X} = N$.

Since μ is adapted and strictly aperiodic, so is μ^K, by [DL_3, corollary 1.4]. Hence $\overline{(I-U^K)X} = N = \overline{(I-U)X}$, and $\|U^n x\| \to 0$ for $x \in N$ by Theorem 2.9.

THEOREM 2.11: *Let μ be a spread-out probability, with support S, on a locally compact σ-compact group G. Then the following are equivalent:*

(i) $e \in \overline{\bigcup_{k=1}^{\infty} \bigcup_{j=0}^{\infty} S^{-j} S^{j+k}}$.
(ii) *For some $K \geq 1$, $\|\mu^{n+K} - \mu^n\| \to 0$.*
(iii) *There exists $K \geq 1$, such that for any bounded continuous representation of G in a Banach space, $\|U^n(I-U^K)\| \underset{n\to\infty}{\to} 0$.*
(iv) *Same as in (iii), but with $U^n(I-U^K) \to 0$ strongly.*
(v) $e \in \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{\infty} S^{-j} S^{j+k}$.

Proof: (ii)⇒(iii)⇒(iv) is obvious, and (iv)⇒(i) follows from Corollary 2.7(iii).

(i)⇒(ii): We look at the canonical representation by right translations in $L_1(G,\lambda)$, but use the general notation.

Fix $x \in L_1(G,\lambda)$ and define $g_{n,k,y}(t) = \|T(t)U^n y + U^{n+k} y\|_1$. We prove the equicontinuity of

$$\{g_{n,k,y}\colon n \geq 0,\quad k \geq 0,\quad y \in C = \{x - U^m x\colon m \geq 0\}\}.$$

Let $\mu^n = \nu_n + \eta_n$ be the Lebesgue decomposition of μ^n. Since μ is spread out, $\nu_{n_0} \neq 0$ for some n_0, so $\|\eta_{n_0}\| < 1$. Hence $\|\eta_{jn_0}\| \leq \|\eta_{n_0}^j\| \leq \|\eta_{n_0}\|^j \to 0$.

Fix $\varepsilon > 0$. There exists N such that $\|\mu^N - \nu_N\| < \varepsilon/6\|x\|$. Since $\nu_N << \lambda$, by continuity of the translations in $L_1(G,\lambda)$ we obtain a neighbourhood A_1 of e such that $\|\delta_t * \nu_N - \nu_N\| < \varepsilon/6\|x\|$ for $t \in A_1$.

For $n \geq N$ and $t^{-1}s \in A_1$ we now obtain

$$\begin{aligned}
&\|T(t)U^n x - T(s)U^n x\| \\
&\qquad \leq \|[T(t) - T(s)](U^N - U_{\nu_N})U^{n-N}x\| + \|[T(t) - T(s)]U_{\nu_N}U^{n-N}x\| \\
&\qquad \leq 2\|\mu^N - \nu_N\|\|x\| + \|(U_{\delta_t * \nu_N} - U_{\delta_s * \nu_N})U^{n-N}x\| \\
&\qquad \leq [2\|\mu^N - \nu_N\| + \|\delta_{t^{-1}s} * \nu_N - \nu_N\|]\|x\| < \varepsilon/2.
\end{aligned}$$

Since the representation is strongly continuous, there exists a neighbourhood A_2 of e such that, for $1 \leq n < N$, $t^{-1}s \in A_2$ implies (since $\|T(t)\| = 1$)

$$\|T(t)U^n x - T(s)U^n x\| = \|T(t^{-1}s)U^n x - U^n x\| < \varepsilon/2.$$

Put $A = A_1 \cap A_2$. Then for $t^{-1}s \in A$, $n \geq 1$ and $y = x - U^m x$, we have

$$\begin{aligned}
|g_{n,k,y}(t) - g_{n,k,y}(s)| &\leq \|T(t)U^n y - T(s)U^n y\| \\
&\leq \|T(t)U^n x - T(s)U^n x\| \\
&\quad + \|T(t)U^{n+m}x - T(s)U^{n+m}x\| < \varepsilon.
\end{aligned}$$

By the equi-continuity of $\{g_{n,k,y}\}$, for $\varepsilon > 0$ there exists an open set $A \ni e$ such that

$$\big|\, \|T(t)U^n y + U^{n+k}y\|_1 - \|U^n y + U^{n+k}y\|_1 \big| = |g_{n,k,y}(t) - g_{n,k,y}(e)| < \varepsilon$$

for every $t \in A$, $n \geq 0$, $k \geq 0$ and $y = x - U^m x$. This implies that for $n \geq 0$, $k \geq 0$ and $y = x - U^m x$, we have

$$\|U^n y + U^{n+k}y\|_1 \geq \|T_t U^n y + U^{n+k}y\|_1 - \varepsilon \quad (t \in A). \tag{2.3}$$

By (i) there exist $K \geq 1$ and $j \geq 0$ such that $(\check{\mu}^j * \mu^{j+K})(A) > 0$. Together with Lemma 2.8 (ii) this implies the existence of a $t_0 \in A$ and $n_k \uparrow \infty$ such that

$$\|T_{t_0}U^{n_k}y + U^{n_k+K}y\|_1 \geq 2 \lim_{n\to\infty} \|U^n y\|_1 - \varepsilon.$$

Combining this with (2.3) we get

$$\|U^{n_k}y + U^{n_k+K}y\|_1 \geq 2 \lim_{n\to\infty} \|U^n y\|_1 - 2\varepsilon,$$

and therefore

$$\lim_{n\to\infty} \|U^n \frac{1}{2}(I + U^K)y\|_1 \geq \lim_{n\to\infty} \|U^n y\|_1 - \varepsilon \quad (y \in C).$$

Choosing now $y = x - U^K x$ we obtain

$$(2.4)\quad \begin{aligned} &\lim_{n\to\infty} \|U^{n+K}x - U^n x\|_1 \le \lim_{n\to\infty} \|U^n \tfrac{1}{2}(I + U^K)(I - U^K)x\|_1 + \varepsilon \\ &= \lim_{n\to\infty} \|U^n \tfrac{1}{2}(I - U^{2K})y\|_1 + \varepsilon \le \lim_{n\to\infty} \|U^n x\|_1 + \varepsilon \le \|x\|_1 + \varepsilon. \end{aligned}$$

Since μ is spread-out, ν_N the absolutely continuous part of μ^N, satisfies $\|\mu^N - \nu_N\| < \varepsilon$ for some N. Let $h = d\nu_N/d\lambda$, and denote by Δ the modular function in G (i.e., $\Delta d\lambda$ is a left Haar measure). For any $f \in L_1(G,\lambda)$, let $\check{f}(s) = f(s^{-1})$. Then we have (see [LW, Lemma 1.1]) $\|\mu^n * \nu_N\| = \|\mu^n * (\Delta\check{h})\|_1$.

We put $x = \Delta\check{h}$ in (2.4) and obtain

$$\begin{aligned} \lim_{n\to\infty} \|\mu^{n+N+K} - \mu^{n+N}\| &= \lim_{n\to\infty} \|(\mu^{n+K} - \mu^n) * \mu^N\| \\ &\le \lim_{n\to\infty} \|\mu^{n+K} * \nu_N - \mu^n * \nu_N\| + 2\varepsilon \\ &= \lim_{n\to\infty} \|U^{n+K}x - U^n x\|_1 + 2 \\ &\le \|x\|_1 + 3\varepsilon = \|\nu_N\| + 3\varepsilon \le 1 + 3\varepsilon < 2, \end{aligned}$$

if we chose $\varepsilon < \frac{1}{3}$ at the beginning. By Foguel's zero-two law [F_2], $\lim_{n\to\infty} \|\mu^{n+K} - \mu^n\| = 0$.

We now have the equivalence of (i), (ii), (iii), (iv), and clearly (v)$\Rightarrow$(i). To complete the proof, we show (ii)$\Rightarrow$(v). By (ii), there is n with $\|\mu^{n+K} - \mu^n\| < 2$. Since $\mu^{n+K}(S^{n+K}) = \mu^n(S^n) = 1$, we have $S^{n+K} \cap S^n \neq \emptyset$. If $t \in S^{n+K} \cap S^n$, then $e = t^{-1}t \in S^{-n}S^{n+K}$, implying (v). ∎

Remarks: (1) A special case of Theorem 2.11 is when μ is irreducible. It is not clear if we have $K = 1$ for μ irreducible and strictly aperiodic.

(2) If μ is not spread-out, the result of the theorem may fail (even for *weak* convergence). Let $G = \mathbb{R}$ and take $\mu = \frac{1}{2}(\delta_{-1} + \delta_{\sqrt{2}})$, which is irreducible, but not spread-out. Let $T(t)$ be induced in $L_2[0,1]$ by $\theta_t\alpha = \alpha + t/(1+\sqrt{2}) \bmod 1$. Then U is an irrational rotation, so $U^{nK}(I - U^K)$ does not converge weakly for any K. Thus, the condition on S in Theorem 2.9 cannot be weakened to that of Theorem 2.11.

(3) For μ adapted, strictly aperiodic and spread-out, the result of the theorem may fail without the condition on e, by Glasner's example [G]. (Since G is discrete, even weak convergence fails).

THEOREM 2.12: *Let μ be an irreducible strictly aperiodic spread-out probability on a locally compact σ-compact group G. For every bounded continuous representation of G in a reflexive space, U^n converges strongly to a projection P on F, with $P(N) = \{0\}$.*

Proof: The same as for Corollary 2.10, except that we use Theorem 2.11. ∎

Remarks: (1) Without strict aperiodicity, Theorem 2.11 yields the strong convergence of U^{nK}, for any bounded continuous representation in a reflexive space. For representations by contractions in a uniformly convex space, this is in [DL3, prop. 1.6 and Theorem 2.9].

(2) It is not known if in Theorem 2.12, instead of irreducible, we can take μ adapted.

We now study the condition on S used in Theorem 2.11. We denote by H the smallest closed normal subgroup, a class of which contains S.

LEMMA 2.13: *If the closed group generated by S is G, and*

$$e \in \overline{\bigcup_{k=1}^{\infty} \bigcup_{j=0}^{\infty} S^{-j} S^{j+k}},$$

then G/H is a compact monothetic Abelian group.

Proof: Let p be the canonical projection of G onto G/H. By definition, $p(S)$ is a single point $p(t)$ (with $t \in S$), and it generated G/H as a closed group [DL3, proposition 1.6], so G/H is monothetic Abelian, either compact — or isomorphic to $\mathbb{Z}$.

For $r \in S^{-j} S^{j+k}$ $(j \geq 0, k \geq 1)$, we have $p(r) = p(t)^k$, so by our assumption (and continuity of p) $e_{G/H} = p(e)$ is in the closure of $\{p(t)^k\}_{k \geq 1}$. Hence G/H cannot be isomorphic to $\mathbb{Z}$. ∎

3. Strong convergence of U^n in uniformly convex spaces

Another important problem in the study of random walks on groups, is that of the convergence to zero, in non-compact groups, of the "concentration function" $\|\mu^n * 1_K\|_\infty$ for K compact. It was shown in [DL3] that (i) of Corollary 2.5 implies (ii). The main result of [DL3] is that if μ is adapted, strictly aperiodic and spread-out, then U_μ^n converges strongly for any continuous representation by isometries on a uniformly convex Banach space. It gives an affirmative answer to the "concentration function" problem for μ spread-out. For more discussion, see [DL3] and [HoM]. Weak convergence was shown in [D1] and in [M].

LEMMA 3.1: *Let $\mathcal{S}$ be a locally compact semi-group and let μ be a probability on $\mathcal{S}$. Let $T(t)$ be a bounded continuous representation of $\mathcal{S}$ in a reflexive Banach space X. If*

$$\lim \|U^n T(t)x - U^n T(s)x\| = 0, \quad x \in X; \quad t, s \in \mathcal{S} \tag{3.1}$$

then U^n converges strongly.

Proof: For $t, s \in \mathcal{S}$ (3.1) yields

$$\|[U^n - U^n T(s)]T(t)x\| = \|U^n T(t)x - U^n T(st)x\| \xrightarrow[n\to\infty]{} 0.$$

Hence, by integrating and using Lebesgue's theorem,

$$\begin{aligned}\|(U^n - U^{n+1})T(t)x\| &= \left\| \int [U^n - U^n T(s)]T(t)x d\mu(s) \right\| \\ &\le \int \|[U^n - U^n T(s)]T(t)x\| d\mu(s) \to 0.\end{aligned}$$

We may assume $X = \text{clm} \bigcup \{T(t)X \colon t \in \mathcal{S}\}$, so we have $U^n(I-U) = U^n - U^{n+1} \to 0$ strongly, and by the ergodic decomposition of the reflexive X, U^n converges strongly. ∎

THEOREM 3.2: *Let $\mathcal{S}$ be a locally compact semi-group and let μ be a probability on $\mathcal{S}$ such that $\mu \times \mu$ is ergodic on $\mathcal{S} \times \mathcal{S}$. Then for every continuous representation of $\mathcal{S}$ by contractions in a Hilbert space, $(U_\mu)^n$ converges strongly to the orthogonal projection on the common fixed points.*

Proof: Let $U = U_\mu$. We show that (3.1) holds. Fix $x \in X$, and let $h_n(s,t) = \langle U^n T(s)x, U^n T(t)x \rangle$. Then clearly

$$2h_n(s,t) = \|U^n[T(t)x + T(s)x]\|^2 - \|U^n T(t)x\|^2 - \|U^n T(s)x\|^2.$$

Since $\|U\| \le 1$, each term on the right hand side converges as $n \to \infty$, so $h(s,t) = \lim h_n(s,t)$ is well defined, and is continuous by equicontinuity of $\{h_n\}$. Since by definition

$$\int\int h_n(ss_0, tt_0) d\mu(s) d\mu(t) = h_{n+1}(s_0, t_0),$$

Lebesgue's theorem yields that $h*(\mu\times\mu)=h$, and by ergodicity of $\mu\times\mu$, $h(s,t)$ is constant on $\mathcal{S}\times\mathcal{S}$. (3.1) follows from

$$\|U^nT(t)x-U^nT(s)x\|^2=h_n(t,t)-2h_n(s,t)+h_n(s,s)\xrightarrow[n\to\infty]{}0.$$

Now we apply Lemma 3.1 to obtain strong convergence of U^n, to a projection on $\{y\colon Uy=y\}$. Let $Uy=y$. The strict convexity yields $T(s)y=y$ for every $s\in S=\mathrm{supp}(\mu)$. Hence for $t\in\mathcal{S}$ we take some $s\in S$ and use (3.1) to obtain

$$\|y-U^nT(t)y\|=\|U^nT(s)y-U^nT(t)y\|\to 0.$$

Since $\|U\|\le 1$, also $U^*y=y$ [K, p.3], so for $t\in\mathcal{S}$ we have

$$\langle y,T(t)y\rangle=\langle U^{*n}y,T(t)y\rangle=\langle y,U^nT(t)y\rangle\to\|y\|^2.$$

Since $\|T(t)\|\le 1$, we must have $T(t)y=y$. ∎

PROPOSITION 3.3: *Let μ be a probability on a locally compact σ-compact group G. Then μ is completely mixing $\Rightarrow$ $\mu\times\mu$ is ergodic $\Rightarrow$ μ is ergodic and strictly aperiodic.*

Proof: For any Markov operator on L_1 of a σ-finite measure space, complete mixing implies ergodicity of the Cartesian square (e.g. [AaLWe]), which clearly implies ergodicity of μ. Ergodicity of $\mu\times\mu$ implies that if $0\not\equiv f\in L_\infty(G)$ with $\mu*f=\lambda f$, $|\lambda|=1$, then $\lambda=1$ (and f is constant), since $(\mu\times\mu)(f\otimes\bar f)=f\otimes\bar f$. The strict aperiodicity follows from the following lemma.

LEMMA 3.4: *Let μ be an adapted probability, with support S, on a locally compact σ-compact group. If μ is not strictly aperiodic, there exists a continuous character f on G, $f=\lambda\ne 1$ on S, $|\lambda|=1$, and $\mu*f=\lambda f$, $\check\mu*f=\bar\lambda f$.*

Proof: Let H be the smallest closed normal subgroup, a class of which contains S. Since μ is adapted, G/H is an Abelian monothetic group [DL$_3$,1.6], and there is $r\in G$ with $S\subset Hr$. We assume μ not strictly aperiodic, i.e., $H\ne G$. Then $r\notin H$, since μ is adapted and G/H is non-trivial.

Let φ be the canonical map of G onto G/H. Then $\varphi(r)\ne e_{G/H}$ since $r\notin H$, and by commutativity there is a continuous character χ on G/H with $\chi(\varphi(r))=\lambda\ne 1$. Define $f(t)=\chi(\varphi(t))$. For $s\in S$ we have $\varphi(s)=\varphi(r)$, so $f(s)=f(r)=\lambda$.

Clearly f is a continuous character, and satisfies

$$\mu * f(t) = \int f(ts)d\mu(s) = \int f(t)f(s)d\mu(s) = \lambda f(t),$$
$$\check{\mu} * f(t) = \int f(ts^{-1})d\mu(s) = \int f(t)\overline{f(s)}d\mu(s) = \bar{\lambda} f(t). \quad \blacksquare$$

Remark: If G is Abelian or compact, then the 3 conditions of Proposition 3.3 are equivalent — see section 2. This is also true if μ is spread-out [G], or recurrent (Theorem 2.2).

LEMMA 3.5: *Let $T(t)$ be a continuous representation of a locally compact semi-group S by contractions in a uniformly convex Banach space X. Then for every $x \in X$ we have $\int \|T(t)U_\mu^n x - U_\mu^{n+1}x\|d\mu(t) \to 0$.*

Proof: Since the result is obvious for $\|U^n x\| \to 0$, assume $\|U^n x\| \downarrow c > 0$. Let $\int \|T(t)U^{n_i}x - U^{n_i+1}x\|d\mu(t)$ converge to a number α. By Lemma 2.6 with $j = 0$, there is a subsequence of $\{n_i\}$ (still denoted by $\{n_i\}$) with

$$\|T(t)U^{n_i}x + U^{n_i+1}x\| \to 2c\mu \quad \text{a.e.}$$

For such a t, let

$$y_i = \frac{T(t)U^{n_i}x}{\|U^{n_i}x\|} \quad \text{and} \quad z_i = \frac{U^{n_i+1}x}{\|U^{n_i}x\|}.$$

Then $\|y_i\| \le 1$, $\|z_i\| \le 1$, and $\|y_i + z_i\| \to 2$. By uniform convexity, $\|y_i - z_i\| \to 0$. Hence $\|T(t)U^{n_i}x - U^{n_i+1}x\| \to 0$ μ a.e., and $\alpha = 0$. ∎

Remark: The lemma is given a different proof in [DL3] (which applies also to semi-groups). A simple proof for Hilbert spaces is also given there.

THEOREM 3.6: *Let G be a locally compact group in [SIN]. Let μ be an adapted and strictly aperiodic probability on G, and let $T(t)$ be a continuous representation of G by contractions in a uniformly convex Banach space. Then U^n converges strongly to a projection P on F, with $P(N) = \{0\}$.*

Proof: Fix $x \in X$. Let $g_n(t) = \|T(t)U^n x - U^{n+1}x\|$. Then $g_n(t)$ is (uniformly) equicontinuous (which is proved as in Theorem 2.9). We show that

$$\|T(t)U^n x - U^{n+1}x\| \to 0 \quad \text{for every } t \in S.$$

Let $t_0 \in S$. Let $\{n_i\}$ be a subsequence with $\lim \|T(t_0)U^{n_i}x - U^{n_i+1}x\| = \alpha$. We show $\alpha = 0$. By Lemma 3.5, there exists a subsequence of $\{n_i\}$ (still denoted

$\{n_i\}$) with $\|T(t)U^{n_i}x = U^{n_i+1}x\| \to 0$ for μ-a.e. t, so

$$S_0 = \{\, t \in S\colon \|T(t)U^{n_i}x - U^{n_i+1}x\| \to 0\}$$

is dense in S. By the equicontinuity of $g_n(t)$, let A be such that $s^{-1}t \in A \Rightarrow |g_n(t) - g_n(s)| < \varepsilon\ \forall n$. There is $t \in S_0$ in t_0A, so

$$g_{n_i}(t_0) \le g_{n_i}(t) + \varepsilon \to 0 + \varepsilon.$$

Since ε is arbitrary, $\alpha = 0$. Hence $\|T(t)U^nx - T(s)U^nx\| \to 0$ for $t, s \in S$ or $\|T(t)U^nx - U^nx\| \to 0$ for $t \in S^{-1}S$. But also $\|U^nx - T(t^{-1})U^{n+1}x\| \to 0$ for $t \in S$, so $\|T(t^{-1})U^nx - T(s^{-1})U^nx\| \to 0$ for $t, s \in S$, or $\|T(t)U^nx - U^nx\| \to 0$ for $t \in SS^{-1}$. Applying the same to U^k, and to the vectors U^jx for $0 \le j < k$, we have $\|T(t)U^nx - U^nx\| \to 0$ for $t \in \bigcup_{j=1}^{\infty}(S^{-j}S^j \bigcup S^jS^{-j})$. Let $\mathcal{S}' = \{\, t \in G\colon \|T(t)U^nx - U^nx\| \to 0\}$. Then $\mathcal{S}'$ is a semi-group (hence a group), dense in G. By the uniform equi continuity, since $\lim \|T(t)U^nx - U^nx\| = 0$ on a dense subset of G, it is 0 everywhere, so $\mathcal{S}' = G$. Now $\|U^{n+1}x - U^nx\| \le \int \|T(t)U^nx - U^nx\| d\mu(t) \to 0$. Hence, by reflexivity, U^n converges strongly. The identification of the limit is proved in Corollary 2.10. ∎

Remark: If μ is spread-out, we do not need to assume G in [SIN], since the equi-continuity of $\{g_n(t)\}$ used in the proof is proved as in Theorem 2.11. We thus obtain a different proof of Theorem 2.8 of [DL$_3$].

THEOREM 3.7: *Let G be a locally compact amenable group, and let μ be a strictly aperiodic adapted probability on G. Then for every bounded continuous representation of G in a Hilbert space, $(U_\mu)^n$ converges weakly to a projection P on F, with $P(N) = 0$.*

Strong convergence to P holds in the following cases:

(i) *G is in [SIN] (In particular, G is discrete, Abelian or compact).*

(ii) *μ is recurrent.*

(iii) *$\mu \times \mu$ is ergodic.*

(iv) *μ is spread-out.*

(v) *μ is symmetric.*

Proof: Since G is amenable, by a result of Dixmier [P, p.187] (or [Ly, p. 83]), the bounded continuous representation $T(t)$ in the Hilbert space is equivalent to a unitary representation. Thus, results for unitary representations transfer to bounded representations. The weak convergence now follows from [DL_3]. Similarly, the strong convergence in the cases (iv) and (v) follows from [DL_3], and in case (iii) — from Theorem 3.2. Case (ii) is a direct consequence of Theorem 2.2 (a recurrent adapted random walk is ergodic) valid for *any* bounded continuous representation. Case (i) follows from Theorem 3.6. ∎

Remark: Each of the conditions (ii)–(iii) implies the needed amenability.

LEMMA 3.8: *Let $\{x_n\}$ be a bounded sequence in a uniformly convex Banach space, satisfying* (i) $\|x_{n+1} + x_{m+1}\| \le \|x_n + x_m\|$ *for* $n, m \in \mathbf{N}$. *If* $\liminf_{n\to\infty} \|x_n - x_{n+k}\| = 0$ *for a given* $k \in \mathbf{N}$, *then* $\|x_n - x_{n+k}\| \xrightarrow[n\to\infty]{} 0$.

Proof: By (i) $\|x_n\|$ decreases, say to α; we have to prove only for $\alpha > 0$.

Let $\varepsilon > 0$. There is n_ε such that $\|x_{n_\varepsilon}\| \le \alpha + \delta$.

Fix $n > n_\varepsilon$. There is $n_i \ge n$ with $\|x_{n_i} - x_{n_i+k}\| \le \delta$. Then by (i)

$$\|x_n + x_{n+k}\| \ge \|x_{n_i} + x_{n_i+k}\| \ge 2\|x_{n_i}\| - \delta \ge 2\alpha - \delta.$$

Since $\|x_{n+k}\| \le \|x_n\| \le \alpha + \delta$, by uniform convexity $\|x_n - x_{n+k}\| < \varepsilon$ (when $\delta > 0$ is chosen according to the uniform convexity). ∎

THEOREM 3.9: *Let $\{x_n\}$ be a bounded sequence in a uniformly convex Banach space, such that*

(i) $\|x_{n+1} + x_{m+1}\| \le \|x_n + x_m\|$ *for* $n, m \in \mathbf{N}$.

(ii) *For every* $\varepsilon > 0$ *there exists* $k \ge 1$ *with* $\liminf_{n\to\infty} \|x_n - x_{n+k}\| < \varepsilon$.

Then either there exists K with $\lim_{n\to\infty} \|x_n - x_{n+K}\| = 0$, *or $\{x_n\}$ has a strongly convergent subsequence.*

Proof: By (i) with $m = n$ we have that $\{\|x_n\|\}$ is decreasing, so let $\alpha = \lim \|x_n\|$, and we have to prove only for $\alpha > 0$.

For $\varepsilon = 1/i$ there is a k_i with $\liminf_{n\to\infty} \|x_n - x_{n+k_i}\| < 1/i$, so we can choose inductively $n_i \uparrow \infty$ with $\|x_{n_i} - x_{n_i+k_i}\| < 1/i$. Hence $\lim_{i\to\infty} \|x_{n_i} - x_{n_i+k_i}\| = 0$.

(a) Assume first that there is an infinite number of values of k_i, so (by taking a subsequence) we may assume $n_i \uparrow \infty, k_i \uparrow \infty, \lim \|x_{n_i} - x_{n_i+k_i}\| = 0$.

Then for any $\delta > 0$ and N, there exists i such that $n_i > N$, and $\|x_{n_i} - x_{n_i+k_i}\| < \delta$, $\|x_{N+k_i}\| < \alpha + \delta$.

Take $\delta_i \downarrow 0$. Let N_1 be such that $\|x_{N_1}\| < \alpha + \delta_1$. Apply the above for $N = N_i$, $\delta = \delta_i$, to obtain $n_{j_i} > N_i$ and k_{j_i} such that $\|x_{n_{j_i}} - x_{n_{j_i}+k_{j_i}}\| < \delta_i$ and $\|x_{N_i+k_{j_i}}\| < \alpha + \delta_i$.

Define now $N_{i+1} = N_i + k_{j_i}$.

By (i), since $n_{j_i} > N_i$ we have

$$\|x_{N_i} + x_{N_{i+1}}\| = \|x_{N_i} + x_{N_i+k_{j_i}}\| \geq \|x_{n_{j_i}} + x_{n_{j_i}+k_i}\| \geq 2\|x_{n_{j_i}}\| - \delta_i \geq 2\alpha - \delta_i.$$

Also $\|x_{N_i}\| \leq \alpha + \delta_i$, $\|x_{N_{i+1}}\| \leq \alpha + \delta_{i+1} < \alpha + \delta_i$, so by uniform convexity (and choice of δ_i)

$$\|x_{N_i} - x_{N_{i+1}}\| < \frac{1}{2^i}.$$

Hence $\{x_{N_i}\}$ is strongly convergent.

(b) Assume now that $\{k_i\}$ takes only finitely many values. Then there exists $k \geq 1$ such that $\liminf_{n\to\infty} \|x_n - x_{n+k}\| = 0$, so, by the previous lemma, $\|x_n - x_{n+k}\| \xrightarrow[n\to\infty]{} 0$. ∎

THEOREM 3.10: *Let G be a locally compact group in [SIN] and let μ be a probability on G with support S, and assume that $e \in \overline{\bigcup_{k=1}^{\infty}\bigcup_{j=0}^{\infty} S^{-j}S^{j+k}}$. If $T(t)$ is a representation of G by contractions in a uniformly convex Banach space X, and $x \in X$, then $\{U^n x\}$ contains a strongly convergent subsequence.*

Proof: The proof of Theorem 2.7 applies to show that for $\varepsilon > 0$ there exists a $k \geq 1$ (chosen so that $\check{\mu}^j * \mu^{j+k}(A) > 0$, which is possible for a neighborhood A of the unit by our assumption), such that for every $y \in C = \operatorname{conv}\{U^m x: m \geq 0\}$ we have

$$\lim \|U^n y + U^{n+k} y\| \geq 2c(y) - \varepsilon.$$

We assume $c(y) > 0$, so for n large $c(y) \leq \|U^{n+k}y\| \leq \|U^n y\| < c(y) + \delta$. By uniform convexity, $\|U^n y - U^{n+k} y\| < \varepsilon$ for large n.

By the previous theorem, either $\{U^n x\}$ contains a convergent subsequence, or for some $k \geq 1$ we have $\lim_{n\to\infty} \|U^n x - U^{n+k} x\| = 0$.

Assume that $\|U^n x - U^{n+k} x\| \xrightarrow[n\to\infty]{} 0$. Let $Y = \operatorname{clm}\{U^n x \mid n \geq 0\}$. Y is invariant under U, and for every $y \in Y$ we have $\|U^n y - U^{n+k} y\| \to 0$. Since Y is reflexive, $Y = \{z \in Y: U^k z = z\} \oplus \overline{(I - U^k)Y}$ by the ergodic decomposition for U^k on Y. Hence $U^{nk} y$ converges strongly on Y, so $U^{nk} x$ converges strongly. ∎

Remarks: (1) If μ is spread-out, a stronger result holds, without assuming G to be in [SIN], by Theorem 2.11.

(2) The assumption on S in Theorem 3.10 seems only slightly weaker than that of Theorem 2.7, but it allows the treatment of μ *irreducible* as a special case.

THEOREM 3.11: *Let G be a locally compact non-compact group in [SIN], and let μ be an adapted probability on G, with support S, such that*

$$e \in \overline{\bigcup_{k=1}^{\infty} \bigcup_{j=0}^{\infty} S^{-j} S^{j+k}}.$$

Then $\|\mu^n * f\|_2 \to 0$ for every $f \in L_2(\lambda)$, and $\|\mu^n * f\|_\infty \to 0$ for every $f \in C_0(G)$.

Proof: We already know, by [D_1] or [M], that $\mu^n * f \to 0$ weakly in L_2, for $f \in L_2$. By the previous theorem we then have $\|\mu^n * f\|_2 \to 0$. The assertion about $C_0(G)$ follows from that in L_2, as was proved in [DL_3]. ∎

THEOREM 3.12: *Let G be a locally compact non-compact group such that there exists an open set V with compact closure, with $t^{-1}Vt = V$ for every $t \in G$. Let μ be a strictly aperiodic adapted or an irreducible probability on G. Then for every compact set K, $\|\mu^n * 1_K(x)\|_\infty \to 0$.*

Proof: Assume first that $G \in [SIN]$, and consider the canonical representation in $L_2(G)$ given by $T(t)f = \delta_t * f$. Since G is not compact, $F = \{0\}$. For μ adapted and strictly aperiodic, theorem 3.6 yields $\|\mu^n * f\|_2 \to 0$ for every $f \in L_2(G)$. This implies the result, by [DL_3].

For μ irreducible, we have $G = \overline{\bigcup_{k=1}^{\infty} S^k}$, so Theorem 3.11 applies.

Let now G be as in the theorem. By [GrMo, 2.5], there exists a compact normal subgroup H such that $G' = G/H \in [SIN]$. Since G is not compact and H is compact, G' is not compact. By the first part of the proof, G' has the required property, and by Remark (4) in [DL_3, p.101], G has the desired property. ∎

Remarks: (1) Our solution to the "concentration function" problem is under two different assumptions: *μ adapted strictly aperiodic*, or *μ irreducible*. The two assumptions are not comparable.

(2) Willis [Wi] has recently completed the analysis of Hoffmann and Mukherjea [HoM], and obtained the convergence to zero of the concentration function, for *μ irreducible*, on *any* locally compact σ-compact group. The general case of μ adapted and strictly aperiodic is still unsolved.

(3) It is known [Pa] that groups satisfying the hypothesis of Theorem 3.12 (this class is denoted by [IN]) are unimodular. (It was noted in [DL_3] that for non-unimodular groups the concentration function tends to zero for any μ adapted strictly aperiodic.)

LEMMA 3.13: *Let G be a locally compact group, and let μ be a probability on G. Let $T(t)$ be a continuous representation of G by contractions in a uniformly convex Banach space X, and $\{x_n\}_{n=0}^{\infty}$ a bounded sequence in X satisfying $U_\mu x_{n+1} = x_n$. Then for $\varepsilon > 0$ there exists N such that for every $n > N$ and $k > 0$,*

$$\|x_{n+k} - U_{\check{\mu}}^k x_n\| < \varepsilon.$$

Proof: Clearly $\|x_n\| = \|U_\mu x_{n+1}\| \leq \|x_{n+1}\|$, so $\lim \|x_n\|$ exists, so by rescaling we may assume $\lim \|x_n\| = 1$ (excluding the trivial $x_n = 0\ \forall n$).

Fix $0 < \varepsilon < 1$. By uniform convexity, there exists $\delta > 0$, such that $\|x + y\| > 2 - \delta$ with $\|x\|, \|y\| \leq 1$ implies $\|x - y\| < \varepsilon/2$. Since $U_\mu^k x_{n+k} = x_n$, we have

$$\begin{aligned} 2\|x_n\| &= \| \int [T(t)x_{n+k} + x_n] d\mu^k(t) \| \leq \int \|T(t)x_{n+k} + x_n\| d\mu^k(t) \\ &= \int \|x_{n+k} + T(t^{-1})x_n\| d\mu^k(t) = \int \|x_{n+k} + T(t)x_n\| d\check{\mu}^k(t) \leq 2\|x_{n+k}\|. \end{aligned}$$

Let $0 < \alpha < \delta/8$ and let $\|x_N\| > 1 - \alpha\varepsilon/2$ (possible since $\|x_n\| \uparrow 1$). Then for $n > N$ and $k > 0$ we obtain

$$\int [2\|x_{n+k}\| - \|x_{n+k} + T(t)x_n\|] d\check{\mu}^k(t) \leq 2(\|x_{n+k}\| - \|x_n\|) < \alpha\varepsilon.$$

Since the integrand is non-negative, we have

$$\check{\mu}^k\Big(\{t\colon 2\|x_{n+k}\| - \|x_{n+k} + T(t)x_n\| > \frac{\delta}{2}\}\Big) \leq \frac{2}{\delta}\alpha\varepsilon$$

Denote the above set by A. If $t \notin A$, then, for $n > N$,

$$\|x_{n+k} + T(t)x_n\| \geq 2\|x_{n+k}\| - \frac{\delta}{2} > 2 - \alpha\varepsilon - \frac{\delta}{2} > 2 - \delta.$$

By the definition of δ, we have that $t \notin A \Rightarrow \|x_{n+k} - T(t)x_n\| < \varepsilon/2$.

Hence, for $n > N$ and $k > 0$, we have

$$\|U_{\check{\mu}}^k x_n - x_{n+k}\| \leq \int \|T(t)x_n - x_{n+k}\| d\check{\mu}^k(t) \leq 2\check{\mu}^k(A) + \frac{\varepsilon}{2} \leq \frac{4\alpha\varepsilon}{\delta} + \frac{\varepsilon}{2} < \varepsilon.$$

THEOREM 3.14: *Let μ be an adapted and strictly aperiodic probability on a locally compact σ-compact group G. Assume that μ is spread-out, or that G is in [SIN]. If $T(t)$ is a continuous representation of G by contractions in a uniformly smooth Banach space X, then U^n converges strongly to a projection P on F, with $P(N) = \{0\}$.*

Proof: By the ergodic decomposition (X is necessarily reflexive) we have to show that $\|\frac{1}{N}\sum_{j=1}^{N} U^j x\| \to 0$ implies $\|U^n x\| \to 0$. Equivalently [D$_1$] we have to show that if $\|x_n^*\| \le 1$ and $U^* x_{n+1}^* = x_n^*$ for $n \ge 0$, then $U^* x_0^* = x_0^*$.

By our assumption, X^* is uniformly convex, and $S(t) = T^*(t^{-1})$ is a continuous representation of G in X^*. Let $V_\mu = \int S(t)d\mu(t)$. Then $U^* = V_{\check{\mu}}$, and by the previous lemma, for $\varepsilon > 0$, $n > N(\varepsilon)$ and $k > 0$, $\|x_{n+k}^* - V_\mu^k x_n^*\| < \varepsilon$.

By Theorem 3.6 for G in $[SIN]$, or by [DL$_3$] if μ is spread-out, we have that for n fixed, $V_\mu^k x_n^*$ converges strongly as $k \to \infty$, say to y_n^*.

Fix $\varepsilon > 0$ and $n > N(\varepsilon)$, and let K be such that $\|V_\mu^k x_n^* - y_n^*\| < \varepsilon$ for $k > K$. Then for $k_1 > K$ and $k_2 > K$,

$$\|x_{n+k_1}^* - x_{n+k_2}^*\| \le \|x_{n+k_1}^* - V_\mu^{k_1} x_n^*\| + \|V_\mu^{k_1} x_n^* - y_n^*\| + \|V_\mu^{k_2} x_n^* - y_n^*\| + \|V_\mu^{k_2} x_n^* - x_{n+k_2}^*\| < 4\varepsilon.$$

Hence $\{x_n^*\}$ is a Cauchy sequence, and let $\lim x_n^* = x^*$. Then

$$\|U^* x_0^* - x_0^*\| = \|U^{*n+1} x_n^* - U^{*n} x_n^*\| \le \|U^* x_n^* - x_n^*\| = \|x_{n-1}^* - x_n^*\| \to 0,$$

so $U^* x_0^* = x_0^*$, and the theorem is proved. ∎

4. Weak convergence of U^n in uniformly convex spaces

The main results of the previous sections assumed ergodicity, and in the case of groups, are applicable only to amenable groups. In this section we extend to semi-groups the weak convergence results obtained for groups in [DL$_3$]. These apply to a *weakly* continuous representation by contractions in a *uniformly* convex space.

THEOREM 4.1: [AlBi] *Let $\mathcal{S}$ be a semi-group of linear contractions in a uniformly convex Banach space X with X^* strictly convex. Then*

(i) $X = \{\, y: Ty = y\ \forall T \in \mathcal{S}\} \oplus \text{clm} \bigcup_{T\in\mathcal{S}} (I - T)X$.

(ii) *For every $x \in X$, $\overline{\text{co}}\{Tx: T \in \mathcal{S}\}$ contains a unique common fixed point, which is Px (P is the projection from* (i) *above).*

Remark: The decomposition in (i) may fail if X^* is not strictly convex. An example in [AlBi] shows that then we may have $x \in X$ with $\overline{\text{co}}\{Tx\} \cap F$ containing two points.

THEOREM 4.2: *Let μ be a probability on a locally compact semi-group $\mathcal{S}$, with support S, such that the closed semi-group generated by $\bigcup_{j=1}^{\infty} S^j(S^j)^{-1}$ is*

S. Then for every weakly continuous representation of S by contractions on a uniformly convex Banach space X, $(U_\mu)^n$ converges weakly to a projection P on F. If X^* is strictly convex, $X = F \oplus N$ and $P(N) = \{0\}$.

Proof: Denote U_μ by U. It was proved in Lemma 3.4 that

$$\lim_{n\to\infty} \int \|T(t)U^n x - U^{n+1}x\| d\mu(t) = 0 \quad \text{for every } x \in X.$$

CLAIM 1: If $\int \|T(t)U^n x - U^{n+1}x\| d\mu(t) \to 0$, then $T(t)U^n x - U^{n+1}x$ converges weakly to 0 for $t \in S$.

Proof: If not, there exist $t_0 \in S$, $x^* \in X^*$, and a subsequence $\{n_i'\}$, such that $\langle x^*, (T(t_0) - U)U^{n_i}x\rangle \to \alpha \neq 0$. Since our assumption implies that $\|(T(t) - U)U^n x\| \to 0$ in μ-measure, there exists a subsequence $\{n_i\} \subset \{n_i'\}$ such that $\|T(t)U^{n_i}x - U^{n_i+1}x\| \to 0$ μ-a.e.

Let $S' = \{t \in S\colon \|T(t)U^{n_i}x - U^{n_i+1}x\| \to 0\}$. Then $\mu(S') = 1$. For every $t \in S'$ we have $\langle x^*, (T(t) - U)U^{n_i}x\rangle \to 0$. Since $t_0 \in S$, for any open set V containing t_0 we have $\mu(V) > 0$, hence $\mu(V \cap S') > 0$. This shows that $t_0 \in \overline{S'}$. By weak continuity (and reflexivity) $T^*(t_0)x^*$ is in the weak closure of $\{T^*(t)x^*\colon t \in S'\}$, so certainly in $\overline{\text{co}}\{T^*(t)x^*\colon t \in S'\}$, which is weakly closed. Since $\{y^* \in X^*\colon \langle y^* - U^*x^*, U^{n_i}x\rangle \to 0\}$ is clearly convex and norm closed, it contains $T^*(t_0)x^*$. Hence

$$0 \neq \alpha = \lim \langle x^*, T(t_0)U^{n_i}x - U^{n_i+1}x\rangle = 0$$

which is a contradiction. ∎

CLAIM 2: If $\int \|T(t)U^n x - U^{n+1}x\| d\mu(t) \to 0$, then $T(t)U^n x - U^n x \xrightarrow{\omega} 0$ for every $t \in SS^{-1}$.

Proof: Let $t \in SS^{-1}$, so there are $t_1, t_2 \in S$ with $tt_1 = t_2$. By Claim 1 (and the continuity of bounded operators also in the weak topology) we have

$$T(t)U^n x - U^n x = T(t)[U^n x - T(t_1)U^{n-1}x] + [T(t_2)U^{n-1}x - U^n x] \xrightarrow{\omega} 0.$$

CLAIM 3: For $x \in X$, the set $S' = \{t \in S\colon T(t)U^n x - U^n x \xrightarrow{\omega} 0\}$ is a closed sub-semi-group.

Proof: It $t_1, t_2 \in S'$, then

$$T(t_1t_2)U^n x - U^n x = T(t_1)[T(t_2)U^n x - U^n x] + [T(t_1)U^n x - U^n x] \xrightarrow{\omega} 0$$

Hence $\mathcal{S}'$ is a semi-group. The proof that it is closed is similar to the proof of Claim 1. ∎

We now conclude the proof of the theorem. Apply Lemma 3.4 and Claim 2 to μ^j, to obtain that $T(t)U^{jn}x - U^{jn}x \xrightarrow{\omega} 0$ for every $t \in S^j(S^j)^{-1}$ and $x \in X$. Substitute $U^k x$, $1 \le k < j$, instead of x, to obtain $T(t)U^n x - U^n x \xrightarrow{\omega} 0$ for every $x \in X$ and $t \in S^j(S^j)^{-1}$. By Claim 3 and our assumption, $\mathcal{S}' = \mathcal{S}$, so $T(t)U^n x - U^n x \xrightarrow{\omega} 0$ for any $t \in \mathcal{S}$ and $x \in X$, which yields $U^n(U-I)x \xrightarrow{\omega} 0$ for every $x \in X$. Since X is reflexive, $X = \{y\colon Uy = y\} \oplus \overline{(I-U)X}$ by the mean ergodic theorem, hence U^n converges weakly to a projection P.

By uniform convexity, $Uy = y \Leftrightarrow T(t)y = y$ for every $t \in S$. Hence $Uy = y \Leftrightarrow T(t)y = y$ for $t \in \bigcup_{j=1}^{\infty} S^j$. If $t \in S^j(S^j)^{-1}$, we have $tt_1 = t_2$ with $t_i \in S^j$, so that $Uy = y$ implies

$$T(t)y = T(t)T(t_1)y = T(t_2)y = y.$$

By the (weak) continuity and our assumption, $y \in F$, so the limiting projection P projects on F. Since $\overline{(I-U)X} \subset N$, $P(N) = \{0\}$ if and only if $F \cap N = \{0\}$. This is satisfied if X^* is strictly convex [AlBi] or if $\mathcal{S}$ is a *group* [RN]. ∎

Remark: If $\mathcal{S}$ is a locally compact group, the condition on the support is satisfied if μ is *irreducible* and strictly aperiodic [DL_3, Proposition 1.2]. The result of [DL_3, Theorem 2.6] is not a corollary of Theorem 4.2, since it applies to μ *adapted* and strictly aperiodic, which may fail to satisfy our assumption. However, the end of our proof identifies the limit in [DL_3].

For Hilbert spaces, we do have a full generalization: μ adapted and strictly aperiodic on a group satisfies the hypotheses of the next theorem.

THEOREM 4.3: *Let μ be a probability on a locally compact semigroup $\mathcal{S}$, with support S, such that the closed semi-group generated by*

$$\bigcup_{j=1}^{\infty} [S^j(S^j)^{-1} \bigcup (S^j)^{-1} S^j]$$

is $\mathcal{S}$. Then for every weakly continuous representation of $\mathcal{S}$ by contractions in a Hilbert space, $(U_\mu)^n$ converges weakly to the orthogonal projection on F.

Proof: The proof of the previous theorem yields that for every $x \in X$ and $t \in S_+ = \bigcup_j S^j(S^j)^{-1}$ we have $(T(t)-I)U^n x \xrightarrow{\omega} 0$. Hence $U^{*n}(T^*(t)-I)x \xrightarrow{\omega} 0$

for $x \in X$ and $t \in S_+$. Since for any contraction T on a Hilbert space we have $T^n x \xrightarrow{\omega} 0 \Leftrightarrow T^{*n}x \xrightarrow{\omega} 0$ by [F3], we obtain $U^n(T^*(t) - I)x \xrightarrow{\omega} 0$ for $x \in X$ and $t \in S_+$. Since for any contraction T on a Hilbert space, T and T^* have the same fixed points [K, p.3], $\overline{(I - T(t))X} = \overline{(I - T^*(t))X}$. Since $\{ y \in X: U^n y \xrightarrow{\omega} 0\}$ is a closed subspace, we now have that $U^n(T(t) - I)x \xrightarrow{\omega} 0$ for $t \in S_+$ and $x \in X$.

Let $\mathcal{S}^*$ be the semi-group with the elements of $\mathcal{S}$, and multiplication operation $s \circ t = ts$, and the same topology. Then $T^*(s \circ t) = T^*(ts) = T^*(s)T^*(t)$, so now $T^*(\cdot)$ is a representation of $\mathcal{S}^*$, and for the probability μ, $U^*x = \int T^*(s)x d\mu(s)$ defines the average of that representation. Now $A \circ B^{-1} = B^{-1}A$ and $B^{-1} \circ A = AB^{-1}$, so $S^j \circ (S^j)^{-1} = (S^j)^{-1}S^j$. ($S^j$ is the same for the multiplication $\circ$.)

We apply our result to the representation $T^*(\cdot)$ of $\mathcal{S}^*$ and obtain that

$$U^{*n}(T^*(t) - I)x \xrightarrow{\omega} 0$$

for every $x \in X$ and $t \in \bigcup_j (S^j)^{-1}S^j = \bigcup S^j \circ (S^j)^{-1}$.

Hence we have also $(T(t) - I)U^n x \xrightarrow{\omega} 0$ for $t \in \bigcup_{j=1}^{\infty}(S^j)(S^j)^{-1}$. Claim 3 of the previous theorem and our assumption yield that $(T(t) - I)U^n x \to 0$ for every $x \in X$ and $t \in S$, so U^n converges weakly, to the orthogonal projection on $\{ y: Uy = y\}$, which is proved to equal F as in the previous theorem (remembering $Uy = y \Leftrightarrow U^*y = y$). ∎

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